The Hopf Fibration: Homotopy Groups of Spheres

Trey Connelly and Kevin Hu

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1 Introduction

This paper is meant as a summary introduction into the idea of homotopy groups of spheres. We will briefly discuss the meaning of homotopy groups and higher-dimensional spheres, and then provide the reader some simple low-dimensional examples.

The main aspect of this paper is the hopf fibration, a special non-trivial map from the sphere embedded in four dimensions to the traditional sphere embedded in three dimensions. Some necessary background information is provided, followed by a discussion of interesting details of the mapping, as well as the applications of the mapping to higher homotopy groups.

This paper is designed for students in an introductory topology course, and can be easily understood with a basic knowledge of the concepts of topological spaces and fundamental groups.

2 Homotopy Groups

In class, we have been familiarized with the idea of a **fundamental group**, which is defined to be the group of homotopy classes of loops in a topological space X based at point p:

$$\pi_1(X, p) = \{[\gamma] : \gamma \text{ is a loop based at p}\}$$

A loop is homotopic to a circle $\mathbb{S}^1$ and π_1 can thus be interpreted as the group of mappings up to homotopy of $f : \mathbb{S}^1 \to X$. By using this alternative definition, the concept of fundamental groups can be extended for "loops" of higher dimensions. We then define:

$$\pi_n(X, p) = \{[\gamma] : \mathbb{S}^n \to X, \gamma(0, ..., 0) = p\}$$

π_n is called the n^{th} homotopy group and can be seen as a n-dimensional analogue of the fundamental group π_1.

3 The n-Sphere ($\mathbb{S}^n$)

3.1 Overview

Before we can explore the concepts of the Hopf Fibration, it is important to have an understanding of the structure and properties of spheres. In particular, we would like to generalize the "ordinary" sphere $\mathbb{S}^2$, which is a 2-dimensional object embedded in $\mathbb{R}^3$, to any dimension. To begin, we can look at the 2-sphere, as well as its low dimension analogues. $\mathbb{S}^2$ is defined as the set of points $(x, y, z) \in \mathbb{R}^3$ of distance 1 from the origin (such that $x^2 + y^2 + z^2 = 1$); similarly, $\mathbb{S}^1$ (the circle) is the set of points

$(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 = 1$; and the 0-dimensional sphere is essentially the points ± 1 on the real number line. Extending this idea to higher dimensions, the n-sphere $\mathbb{S}^n$ can then be defined as:

$$\mathbb{S}^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

Although n-spheres are difficult to visualize in higher dimensions, they can be understood intuitively as the set of points $(x_1, ..., x_{n+1})$ that are one unit away from the origin.

3.2 Stereographic Projection

It is often useful in topology to interpret an object in various ways, and for the purpose of deriving the hopf fibration, we will outline the projection of the n-sphere into $\mathbb{R}^n$, called the stereographic projection. For instance, we provide an example of the stereographic projection of the 2-sphere onto the plane $\mathbb{R}^2$. Let $N(0, 0, 1)$ be the north pole of the sphere. Define $\sigma_2 : \mathbb{S}^2 - \{N\} \to \mathbb{R}^2$ to be the mapping that sends a point $p \in \mathbb{S}^2, p \neq N$ to the intersection of the line containing N and p with the plane $z = 0$. This mapping is given explicitly by:

$$\sigma_2(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

And the inverse mapping is given by:

$$\sigma_2^{-1}(x_0, y_0) = \left(\frac{2x_0}{{x_0}^2 + {y_0}^2 + 1}, \frac{2y_0}{{x_0}^2 + {y_0}^2 + 1}, \frac{{x_0}^2 + {y_0}^2 - 1}{{x_0}^2 + {y_0}^2 + 1}\right)$$

The derivation involves dull calculations involving coordinate geometry and will not be shown. However, we know that there exists a unique line that passes through N and p and that the line will always intersect the plane $z = 0$ exactly once, so σ is well-defined. It is also quite easy to check its bijectivity and the proof can be left as an exercise for the reader.

Perhaps the most important takeaway from this mapping is that it demonstrates a homeomorphism between the 2-sphere minus a point and the real plane, which tells us that the sphere behaves very similarly to a plane. Another way to convey the mapping is to include the north pole in the domain, but at the same time add a **point at infinity** to the codomain ($\sigma_2 : \mathbb{S}^2 \to \mathbb{R}^2 + \{\infty\}$). By definition of the mapping, if you take p to be equal to N, the line containing p and N thus becomes a line that is parallel to the plane $z = 0$, so we can think of the intersection between the line and the plane as a point at infinity.

In general, we can extend the function above to the stereographic projection of the n-sphere $\sigma : \mathbb{S}^n - \{N(0, ..., 1)\} \to \mathbb{R}^n$. defined by:

$$\sigma_n(x_1, x_2, ..., x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, ..., \frac{x_n}{1 - x_{n+1}}\right)$$

with an inverse of:

$$\sigma_n^{-1}(y_1, y_2, ..., y_n) = \frac{1}{\sum\limits_{i=1}^{n} y_i^2 + 1} \left(2y_1, 2y_2, ..., 2y_n, \sum_{i=1}^{n} y_i^2 - 1\right)$$

Another interesting fact about the stereographic projection that is perhaps worth mentioning is that the mapping preserves angles (*conformal*) and circles. That is, given a circle or an angle between two lines on the sphere, the image of the circle is also a circle and the image of the lines would intersect at the same angle.

4 Homotopy of Spheres

Now let us examine the homotopy groups of spheres, $\pi_k(\mathbb{S}^n)$.

First, let us examine the case $k < n$. There are no nontrivial continuous maps from a lower-dimensional sphere to a higher-dimensional sphere. All such maps are either non-surjective or can be deformed to be so, and can thus be considered maps to a punctured sphere. As explained above, the punctured n-sphere is homeomorphic to real n-space through stereographic projection, and is thus contractible. Therefore, any such mapping can be homotoped to a point, so is trivial, yielding $\pi_k(\mathbb{S}^n) = 0$.

Next, the case $k = n$ comes as a consequence of the Hurewicz theorem. Because the n-sphere is path-connected, the first nonzero homotopy group is isomorphic to the first nonzero homology group. The first and only nonzero homology group is $H_n(\mathbb{S}_n) = \mathbb{Z}$, which then implies that $\pi_n(\mathbb{S}_s) = \mathbb{Z}$.

The interesting part of homotopy groups is the case $k > n$. The higher homology groups of spheres are all trivial, but this is not always, or often, true for higher homotopy groups. The only trivial case is that of $\mathbb{S}^1$: for any $k > 1$, $\pi_k(\mathbb{S}^1) = 0$. This results from the 1-sphere being covered by the contractible real line: because the k-sphere is simply connected, any map to $\mathbb{S}^1$ can be lifted to the real line and then homotoped to a point.

Unfortunately, higher-dimensional spheres do not have contractible covers and thus do not have trivial higher homotopy groups. The first such example is the case $\pi_3(\mathbb{S}^2)$, the homotopy group of maps from the 3-sphere to the 2-sphere. We will now begin to introduce the hopf fibration, a nontrivial mapping that will generate this homotopy group.

5 Fiber Bundles: a Brief Overview

In order to understand the hopf fibration, it is important to first briefly explain a few key concepts. First is the idea of a *fiber bundle*.

A fiber bundle is a structure consisting of three spaces (F, E, B) and a continuous surjective map $p : E \to B$. We require p to be *locally trivial*, in that for every point $x \in B$, there exists a neighborhood $U \subset B$ containing x such that the preimage of U is homeomorphic to $U \times F$. We call E the *total space*, B the *base space*, and F the *fiber*. The bundle can be denoted denoted $F \hookrightarrow E \xrightarrow{p} B$.

The essential structure of a fiber bundle is a map in which the preimage of every point in the base space is homeomorphic to the fiber. A fibration is a looser form of a fiber bundle in that the preimages, or fibers, of points in the base space must only be homotopy equivalent, not necessarily homeomorphic.

6 Quaternions

In order to define the hopf fibration, we must introduce, quaternions, an extension of the complex numbers. Complex numbers extend the real numbers by combining two real coordinates (a,b) in the form $a + bi$. 1 and i form a vector basis of $\mathbb{R}^2$, satisfying the relationship $i^2 = -1$. Similarly, quaternions extend complex numbers using a basis $1, i, j, k$ of $\mathbb{R}^4$. Four real coordinates (a, b, c, d) are combined to form a quaternion $a + ib + jc + kd$, and the basis elements satisfy the properties $i^2 = j^2 = k^2 = ijk = -1$. From these relations we can derive an understanding of quaternion multiplication, which, while not commutative, is well-defined.

Quaternions can be used to define rotations in $\mathbb{R}^3$: the space is rotated about the origin by a vector (b, c, d) in $\mathbb{R}^3$ and an angle $2\arccos(a)$. The explicit rotation for a quaternion r is a linear mapping $R_r : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $R_r(p) = rp\bar{r}$. In this map, $p = (p_1, p_2, p_3)$, a point in $\mathbb{R}^3$ is associated to the *pure* quaternion (having no real part) $p_1 i + p_2 j + p_3 k$, and then conjugated by r under quaternion multiplication. This will always result in another pure quaternion, which is then associated to the corresponding point in $\mathbb{R}^3$.

Because only the direction and not the magnitude of the quaternion vector is used in rotations, we can restrict r to the set of unit quaternions defined by $a^2 + b^2 + c^2 + d^2 = 1$, which clearly forms the 3-sphere, $\mathbb{S}^3$. Only antipodal points correspond to the same rotation, so the quotient set by equating antipodal points forms the 3D rotation group SO(3), the group of rotations around the origin in $\mathbb{R}^3$ under composition.

7 The Hopf Fibration

7.1 A map in three forms

Now we can define the Hopf fibration using quaternions. For $P_0 = (1, 0, 0)$,

$$h(r) = R_r(P_0) = ri\bar{r} \tag{1}$$

The fibration takes a unit quaternion in $\mathbb{S}^3$ and uses it to perform a rotation on P_0. The rotation will preserve the length of P_0, and thus send it to another point on the sphere. We can calculate the real equivalent $h : \mathbb{S}^3 \to \mathbb{S}^2$ by assigning each unit quaternion to its corresponding point in $\mathbb{R}^3$:

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad - bc), 2(bd - ac)) \tag{2}$$

The simplest equivalent mapping is from the unit sphere in $\mathbb{C}^2$, which is homeomorhpic to $\mathbb{S}^3$, to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which is homeomorphic to $\mathbb{S}^2$ through stereographic projection.

$$h(z_1, z_2) = z_1/z_2 \tag{3}$$

This also defines a quotient set on $\mathbb{S}^3$ in $\mathbb{C}^2$ by the equivalence relation $(z_1, z_2) \sim (w_1, w_2)$ if $(w_1, w_2) = (\lambda z_1, \lambda z_2)$ for some λ in $\mathbb{C}$ with $|\lambda| = 1$.

7.2 Fiber structure

We can see from the equivalence relation above that the preimage of a point z_0 is a set of the form $\{(\lambda z_1, \lambda z_2) : \lambda = e^{i\theta}\}$. This is a circle in $\mathbb{S}^3$; in fact, it is a great circle, as it connects antipodal points (z_1, z_2) and $(-z_1, -z_2)$. Thus, the preimage of every point is homeomorphic to $\mathbb{S}^1$. This defines a fiber bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$.

As in the general case, for every point p there exists a neighborhood U with preimage in $\mathbb{S}^3$ homeomorphic to $U \times \mathbb{S}^1$. Particularly of note, a circle of latitude in $\mathbb{S}^2$ (homeomorphic to $\mathbb{S}^1$) has preimage homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, a torus.

Under stereographic projection onto $\mathbb{R}^3$, the fiber of $(1, 0, 0)$ forms the x-axis and the fiber of $(-1, 0, 0)$ forms the unit circle in the yz-plane. Every other point has a fiber which forms a Villarceau circle of a torus (essentially one part of a diagonal cross-section). These tori are both nested and space-filling when projected. Another important and interesting aspect of the fibers is that all projected fiber circles are linked- that is, each circle intersects the plane of each other circle exactly twice, once inside the circle and once outside.

8 Application to Homotopy Groups

8.1 $\pi_3(\mathbb{S}^2)$

The hopf fibration can be used to generate a set of homotopy classes of maps from $\mathbb{S}^3$ to $\mathbb{S}^2$. Instead of each quaternion rotating P_0 by an angle θ equal to $2\arccos(a)$, maps rotating by $n\theta$ for any $n \in \mathbb{Z}$ are also continuous and form a unique homotopy class. The hopf fibration is thus a generator for the third homotopy group of $\mathbb{S}^2$.

As it turns out, there are no other nontrivial continuous mappings (up to homotopy) from $\mathbb{S}^3$ to $\mathbb{S}^2$, so $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

8.2 Higher homotopy groups of the 2-sphere and 3-sphere

In order to understand higher-homotopy relations of the two spheres, we must now introduce the concept of an *exact sequence*. An exact sequence is a sequence of maps between a sequence of spaces, such that the image of one map is equal to the kernel of the next map.

$$\cdots \to A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \to \cdots \tag{4}$$

The fiber bundle can be viewed as a *short exact sequence*, consisting of three spaces. The hopf fibration's sequence is the following:

$$0 \to \mathbb{S}^1 \to \mathbb{S}^3 \xrightarrow{h} \mathbb{S}^2 \to 0 \tag{5}$$

This correctly implies that the inclusion map is injective and the hopf map is bijective. More interestingly, a fibration creates a an infinite, or *long*, exact sequence of the homotopy groups of the three spaces:

$$\cdots \to \pi_k(\mathbb{S}^1) \to \pi_k(\mathbb{S}^3) \to \pi_k(\mathbb{S}^2) \to \pi_{k-1}(\mathbb{S}^1) \to \cdots \to \pi_0(\mathbb{S}^1) \to \pi_0(\mathbb{S}^3) \tag{6}$$

The homomorphisms $\pi_k(\mathbb{S}^1) \to \pi_k(\mathbb{S}^3)$ and $\pi_k(\mathbb{S}^3) \to \pi_k(\mathbb{S}^2)$ are those induced by the inclusion map and the hopf map. The third homomorphism can be properly defined to make the sequence exact, but this is very complicated and will not be shown here.

Because every homotopy group of $\mathbb{S}^1$ is trivial, we can divide the long sequence into subsequences $0 \to \pi_k(\mathbb{S}^3) \to \pi_k(\mathbb{S}^2) \to 0$ for all $k > 2$, which yields

$$\pi_k(\mathbb{S}^3) \cong \pi_k(\mathbb{S}^2) \tag{7}$$

Clearly the second homotopy groups of these spheres differ, as $\pi_2(\mathbb{S}^2) = \mathbb{Z}$, whereas $\pi_2(\mathbb{S}^3)$ is trivial. However, the hopf fibration is able to tell us that all higher homotopy groups of the three-sphere and the two-sphere are equal. This is a very significant result for the homotopy groups of spheres! While it does not tell us what those homotopy groups are (this is an ongoing problem in mathematics), it does guarantee that they will be the same for both spheres.

8.3 Generalizations of the hopf fibration

The standard hopf fibration can be generalized as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n \tag{8}$$

For $n = 1$, this is the fibration discussed above, as $\mathbb{CP}^1$ is the Riemann sphere, which is equivalent to the two-sphere in $\mathbb{R}^3$.

Similarly we can examine $\mathbb{H}^n$, quaternionic space, defined in a similar way to $\mathbb{R}^n$ and $\mathbb{C}^n$, yielding

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{HP}^n \tag{9}$$

The $n = 1$ case again yields a fibration of spheres, now $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$. It can be obtained by dividing two unit quaternions, just as the base hopf map was defined by dividing two unit complex numbers. The exact sequence generated by this fibration yields a slightly more complicated yet still useful result:

$$\pi_n(\mathbb{S}^4) = \pi_n(\mathbb{S}^7) \times \pi_{n-1}(\mathbb{S}^3) \tag{10}$$

Likewise, a fibration can be created from $\mathbb{S}^{8n+7}$ to $\mathbb{OP}^n$, octonionic projective space, with fibers $\mathbb{S}^7$. Once again, for the $n = 1$ case, we can construct a fibration by dividing unit octionions, yielding $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \rightarrow \mathbb{S}^8$, which yields the relation $\pi_n(\mathbb{S}^8) = \pi_n(\mathbb{S}^{15}) \times \pi_{n-1}(\mathbb{S}^7)$

Interestingly, there are no more higher-dimensional fibrations entirely of spheres. This results from $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ being the only *normed division algebras* over the real numbers. This essentially means that division is well-defined, so the hopf map $h(x_1, x_2) = x_1/x_2$ is a well-defined, continuous fibration of spheres.

9 Conclusions

This paper has provided a brief explanation of homotopy groups and the hopf fibration in order to introduce the complex problem that is the computation of homotopy groups of spheres, an active area of research in algebraic topology. Readers are encouraged to further investigate aspects of these homotopy groups, including the stable homotopy groups and spectral sequence computation.

References

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