Path Integrals an Introduction

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Abstract

Here we discuss the path integral formalism for quantization of fields. The basic idea is reviewed and explained. This is completely based on the book "Quantum Field Theory A Modern Introduction" by Michio Kaku. For calculation natural system of units is taken.

1 Introduction

Why do we need path integrals? Consider that a particle is made to move from one point to another. In classical mechanics we try to find the action **S**, such that the variation in the quantity $\delta\mathbf{S}$ is made minimum. That is, in a classical sense the particle will move along that path where the action is minimum. But in the case of quantum mechanics that is not the case. To understand what happens in the quantum world let us go through the double slit experiment.

2 Double slit experiment

Consider double slit experiment with electrons. Due to the wave nature of electron we expect an interference pattern as similar to that of electromagnetic waves with slit width approximately equal to the de Broglie wave length of the particle. Now consider electrons as simple particle moving from the source to the screen. The question is, what all ways are possible for the electron? Perhaps we need to discuss the probability of each path.

Here in this experiment we can see that the electron can choose any of the possible path to reach the screen. Also since the particle is no longer classical it can reach the screen similar to as a photon. The point is, either of the path are equally probable. The total probability that it choose one path is the product of all probability associated with that path. Lets modify the arrangement. Let us put **N** number of such screens and let there be **N** numbers of slit in each screen. What is the transition probability as $\mathbf{N}\rightarrow\infty$?Obviously the particle will reach there with out any problem. But which path does the particle choose? So the first question is what is the probability of finding the particle at a volume element $d\tau$.

$$|\Psi(x,y,z,t)|^2\,d\tau = (\text{Probability of finding the particle between V and } (\text{V}+\text{d}\tau)) \tag{1}$$

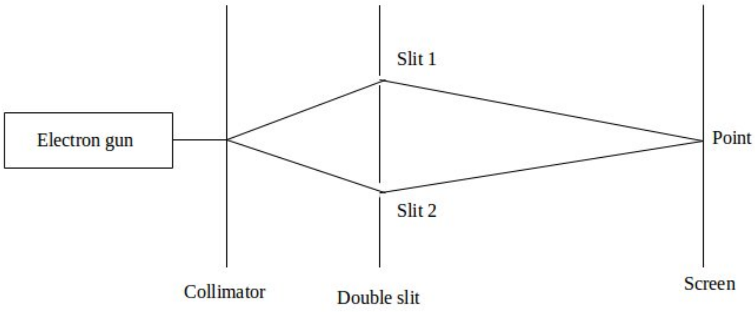

Figure 1: Double slit experiment

In a similar way we can do it in the case of the infinite slits and infinite screens experiment and we will get the total probability of path as follows.

$$P_{a\to b} = \lim_{N\to\infty} \int\int ... \int \prod_{i=1}^{N} |\Psi_i|^2 d\tau_i = 1 \tag{2}$$

This is the fundamental idea. Now let us discuss the postulates of path integrals.

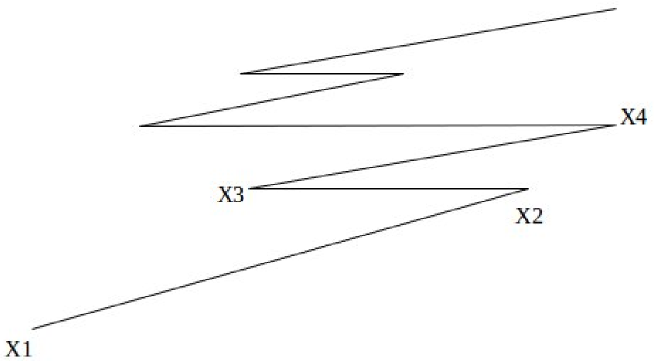

Figure 2: Breaking of path to discrete number of points

Postulate I

The probability P(b,a) of a particle moving from point a to b is the square of the absolute value of a complex number, the transition function K(b,a).

$$P(b, a) = |K(b, a)|^2 \tag{3}$$

Postulate II

The transition function K(b,a) is given by the sum of a phase factor $e^{\frac{iS}{\hbar}}$, where S is the action, taken over all possible paths from $a \to b$:

$$K(b, a) = \sum_{paths} k e^{\frac{iS}{\hbar}} \tag{4}$$

where summation over all paths can be expressed as following.

$$\sum_{paths} = \lim_{N\to\infty} \prod_{i=1}^{3} \prod_{n=1}^{N} \delta x_n^i \to \int Dx \tag{5}$$

The integral $\int Dx$ is not an ordinary integral. It is actually an infinite product of integrals, taken over all possible $dx(t)$. Thus we have

$$K(b,a) = k \int_a^b Dx e^{\frac{iS}{\hbar}} \tag{6}$$

So now we have seen the basic idea behind the path integral formulation. To get a real idea we have to some do some problems.

3 Non-relativistic particle

Consider a free non relativistic point particle in the first quantized formalism. Since the particles is free the Lagrangian will not have a potential term, it will be the kinetic energy of the particle. The action can then be defined as the following.

$$S = \int \frac{1}{2} m \dot{x}_i^2 dt \tag{7}$$

Now instead of the integral, consider the summation. With time interval ϵ. Then the Lagrangian can be written as.

$$dt \to \epsilon \tag{8}$$

$$\frac{1}{2} m \dot{x}_i^2 \to \frac{1}{2} m (x_n - x_{n+1})_i^2 \epsilon^{-1} \tag{9}$$

Then the transition function can be written as the path integral over e^{iS} (Natural system is used where $\hbar = 1$ and $c = 1$).

$$K(b,a) = \lim_{\epsilon\to 0} \int ... \int dx_2 dx_3 ... dx_{N-1} k \ \exp\left[\frac{im}{2\epsilon} \sum_{n=1}^{N-1} (x_n - x_{n+1})^2\right] \tag{10}$$

Now I must say about a major drawback of path integral formalism. One cannot integrate all kind of functions in this method. Only some function are integrable. Gaussian path integral is one which can be integrated. The function has the general form as.

$$\int_{-\infty}^{+\infty} x^{2n} e^{-r^2 x^2} dx = \frac{\Gamma(n+\frac{1}{2})}{r^{2n+1}} \tag{11}$$

Now considering the a part of the path integral we have the following result.

$$\int_{-\infty}^{+\infty} \exp\left[-a(x_1-x_2)^2 - a(x_2-x_3)^2\right] dx_2 = \sqrt{\frac{\pi}{2a}} \exp\left[-\frac{1}{2} a (x_1 - x_3)^2\right] \tag{12}$$

By continuing this process over all the dx_i's we get the following.

$$\int_{-\infty}^{+\infty} \exp\left[-a(x_1-x_2)^2 - ... - a(x_{N-1}-x_N)^2\right] dx_2 ... dx_{N-1} \tag{13}$$

$$= \sqrt{\frac{\pi^{N-2}}{(N-1)a^{N-2}}} \exp\left[-\frac{a}{N-1}(x_1 - x_N)^2\right] \tag{14}$$

from this we can calculate the value of k as.

$$k = \left(\frac{2\pi i\epsilon}{m}\right)^{-\frac{1}{2}N} \tag{15}$$

Now this is just the result of integration, we also need to take the limit $N \to \infty$. Taking that limit we get the following equation for transition function.

$$K(b,a) = \left|\frac{m}{2\pi(t_b - t_a)}\right|^{\frac{1}{2}} \exp\left[\frac{\frac{1}{2}im(x_b - x_a)^2}{t_b - t_a}\right] \tag{16}$$

This can be identified as the Green's function in quantum mechanics.

4 Schrödinger equation

Now we can extend the idea of path integral in to wave function and try to derive the Schrödinger wave equation.

In path integral approach, the evolution of a state is given by the transition function K(b,a). From a classical point of view, this can be viewed as the analogue of Huygen's principle (wave mechanics). This follows that the wave function can be expressed as.

$$\Psi(x_j, t_j) = \int_{-\infty}^{+\infty} K(x_j, t_j; x_i, t_i)\Psi(x_i, t_i)dx_i \tag{17}$$

Now to calculate the time evolution of the wave from $t \to t + \delta t$. We compute the following.

$$\Psi(x, t+\epsilon) = \int_{-\infty}^{+\infty} A^{-1} \exp\left(\frac{im(x-y)^2}{2\epsilon}\right)\Psi(y,t)dy \tag{18}$$

where A is same as the constant k.

$$A = \left(\frac{2\pi i\epsilon}{m}\right)^{-\frac{1}{2}} \tag{19}$$

Now to integrate we make a change of variable as $\eta = y - x$ thus $dy \to d\eta$.

$$\Psi(x, t+\epsilon) = \int_{-\infty}^{+\infty} A^{-1} \exp\left(\frac{im\eta^2}{2\epsilon}\right)\Psi(x+\eta,t)d\eta \tag{20}$$

Now Taylor expansion is done on both sides we get the following equation which can be easily integrated.

$$\Psi(x, t+\epsilon) + \epsilon\frac{\partial\Psi}{\partial t} = \int_{-\infty}^{+\infty} A^{-1} \exp\left(\frac{im\eta^2}{2\epsilon}\right)\left(\Psi(x,t) + \eta\frac{\partial\Psi}{\partial x} + \frac{1}{2}\eta^2\frac{\partial^2\Psi}{\partial^2 x} + ...\right)d\eta \tag{21}$$

The integration over η can be done easily. The integration over the linear term in η vanishes because it is linear, and the integration over the higher term vanish in the limit $\epsilon \to 0$. This gives us.

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\Psi}{\partial^2 x^2} \tag{22}$$

This is Schrödinger equation in the NSU.

References

[1] Michio Kaku, Quantum Field Theory a Modern Introduction, Oxford University Press, New York 1993. page 261-273