On Proving the Identity of the Product Integral

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Let one consider the product integral

$$\int x^{dx} - 1$$

To clarify that the integral extends over to the -1, brackets can be put around the integrand,

$$\int \left[x^{dx} - 1\right]$$

To solve this integral, one may consider the Riemann sum formulated as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

It can be thus stated that

$$\lim_{n\to\infty} \Delta x = dx$$

and that

$$\lim_{n\to\infty} dx = 0$$

Henceforth it can be said that

$$\lim_{\Delta x\to 0} \Delta x = dx$$

Returning to the original integral. the integrand can be multiplied by $\frac{dx}{dx}$ such that the integral is shown by

$$\int \frac{x^{dx} - 1}{dx}\,dx$$

To further simplify the problem, the following function can be taken into account,

$$y = a^x$$

with a being some arbitrary term. To differentiate this function, the natural logarithm of both sides can be taken such that

$$\ln y = \ln\left(a^x\right)$$

Using the property of the natural log function

$$\ln(a^b) = b \ln a$$

the function $\ln y = \ln\left(a^x\right)$ can be expressed as

$$\ln y = x \ln a$$

By differentiating both sides one gets

$$\frac{d}{dx}\left(\ln y\right) = \frac{d}{dx}\left(x \ln a\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

Multiple both sides by y,

$$\frac{dy}{dx} = a^x \ln a$$

Derivatives are also defined by the difference quotient represented as

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f\left(x+h\right) - f(x)}{h}$$

or also

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f\left(x+\Delta x\right) - f(x)}{\Delta x}$$

Applying this to $f(x) = a^x$, one gets

$$\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

a^x can be then factored and pulled out of the limit,

$$\frac{d}{dx} a^x = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

As stated before, $\lim_{n \to \infty} \Delta x = dx$, and thus

$$\frac{d}{dx} a^x = a^x \frac{a^{dx} - 1}{dx}$$

Since it has been proved that

$$\frac{d}{dx} a^x = a^x \ln a$$

one can set the two derivatives equal, such that

$$a^x \ln a = a^x \frac{a^{dx} - 1}{dx}$$

Divide both sides by a^x,

$$\ln a = \frac{a^{dx} - 1}{dx}$$

If a is substituted with x, one is left with

$$\ln x = \frac{x^{dx} - 1}{dx}$$

This is the integrand from the product integral $\int \frac{x^{dx}-1}{dx}\, dx$. By plugging this into the integral, it can be stated that

$$\int \frac{x^{dx} - 1}{dx}\, dx = \int \ln x \, dx$$

and thus

$$\int x^{dx} - 1 = \int \ln x \, dx$$

All that is left now is integration by parts. The following substitutions can be set,

$$u = \ln x$$

$$dv = 1 \, dx$$

The derivative of the natural logarithm of x is $\frac{1}{x}$, and the integral of $1\, dx$ is x, so therefore

$$du = \frac{1}{x}$$

$$v = x$$

This can be substituted for integration by parts, and the integral can now be represented as

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x}\, dx$$

$$\int \ln x \, dx = x \ln x - \int 1 \, dx$$

The integral of $1\, dx$ is once again x,

$$\int \ln x \, dx = x \ln x - x$$

Therefore, it has been proved that

$$\int x^{dx} - 1 = x \ln x - x$$