The Two Snow Plows
Group Problem 2E

May 2, 2016

The Problem

Suppose it starts snowing at a constant rate at 12:00 pm and two snowplows are to be dispatched to clear a long road from the same garage. If Plow X leaves at 1:00 pm and Plow Y leaves at 2:00 pm, will they collide? If so, when? Assume the snowplows can clear snow at a rate inversely proportional to the depth of snow.

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Since the rate of snowfall is constant, we symbolize this with the constant r.

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Substituting:

$$\begin{aligned} \frac{dx}{dt} &= \frac{A}{t}, \\ \frac{dy}{dt} &= \frac{A}{(t-T)}. \end{aligned}$$

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This yields our position function for Plow X:

$$x(t) = A\ln|t|.$$

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This yields Plow Y's position function:

$$t(y) = e^{By}(2 - By).$$

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Plow Y will collide with Plow X e hours after it begins snowing, or approximately 2:43 PM.

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Plow Y will collide with Plow X e^2 hours after it begins snowing, or approximately 7:23 PM.