# Hecke groups, linear recurrences, and Kepler limits

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**Abstract**

We study the linear fractional transformations in the Hecke group $G(\Phi)$ where $\Phi$ is either root of $x^2 - x - 1$ (the larger root being the "golden ratio" $\phi = 2\cos\frac{\pi}{5}$.) Let $g \in G(\Phi)$ and let $z$ be a generic element of the upper half-plane. Exploiting the fact that $\Phi^2 = \Phi - 1$, we find that $g(z)$ is a quotient of linear polynomials in $z$ such that the coefficients of $z^1$ and $z^0$ in the numerator and denominator of $g(z)$ appear themselves to be linear polynomials in $\Phi$ with coefficients that are certain multiples of Fibonacci numbers. We make somewhat less detailed observations along similar lines about the functions in $G(2\cos\frac{\pi}{k})$ for $k \geq 5$.

## 1 Introduction

Let $G(\lambda)$ be the Hecke group generated by the linear fractional transformations $S\colon z \mapsto -1/z$ and $T_\lambda\colon z \mapsto z + \lambda$ and let $G_k = G(2\cos\frac{\pi}{k})$. This article describes numerical experiments carried out to study Hecke groups, mainly $G_k$ for $k \geq 5$. In this article, an $n$-tuple of symbols

$$\overrightarrow{w} = \{w_1, w_2, w_3, ..., w_n\}$$

representing an ordered set of integers is called a *word* on $\mathbf{Z}$ and we write $n = |\overrightarrow{w}|$. A typical element of $G(\lambda)$ takes the form

$$T_\lambda^{w_1} S T_\lambda^{w_2} S ... S T_\lambda^{w_n} = \text{(say) } g_{\lambda;\overrightarrow{w}}.$$

This representation is not unique. For example, a function $g \in G(\lambda)$ can be described by a word of length $n$ for arbitrarily large $n$, because any word representing $g$ can be padded with zeros and the resulting word will also represent $g$. Consequently, when studying all $g$ represented by words $\overrightarrow{w}$ with $|\overrightarrow{w}| \leq N$, we can restrict attention to the words satisfying $|\overrightarrow{w}| = N$.

Let $\phi, \phi^*$, represent the larger and smaller roots of $x^2 - x - 1$, respectively. The problem of expressing (for $g \in G_5$) $g = g_{\phi;\overrightarrow{w}}$ in terms of the $w_i$ was raised by Leo in [9] and discussed by his student Sherkat in [12]; the first purpose of this article is to write down conjectures addressing this question. Our calculations indicate that, for arbitrary $\lambda, z \in \mathbf{C}$, $g_{\lambda;\overrightarrow{w}}(z)$ is a rational function of $z$ and $\lambda$ in polynomials of $\lambda$-degree $\leq k$. Here are the first few:

$$g_{\lambda;\{w_1\}}(z) = \frac{1 \cdot z + w_1\lambda}{0 \cdot z + 1},$$

$$g_{\lambda;\{w_1,w_2\}}(z) = \frac{w_1\lambda \cdot z + w_1w_2\lambda^2 - 1}{1 \cdot z + w_2\lambda},$$

and

$$g_{\lambda;\{w_1,w_2,w_3\}}(z) = \frac{(w_1w_2\lambda^2 - 1) \cdot z + w_1w_2w_3\lambda^3 - (w_1 + w_3)\lambda}{w_2\lambda \cdot z + w_2w_3\lambda^2 - 1}.$$

Following [12], we simplify the above expressions for $g_{\lambda;\overrightarrow{w}}$ when $\lambda = \phi$ or $\phi^*$ by repeatedly making the substitution $\Phi^2 = \Phi + 1 (\Phi = \phi$ or $\phi^*.)$ The coefficients in $g_{\Phi;\overrightarrow{w}}$ become linear polynomials in $\Phi$:

$$g_{\Phi;\{w_1\}}(z) = \frac{1 \cdot z + w_1\Phi}{0 \cdot z + 1},$$

$$g_{\Phi;\{w_1,w_2\}}(z) = \frac{w_1\Phi \cdot z + w_1w_2\Phi + w_1w_2 - 1}{1 \cdot z + w_2\Phi},$$

and

$$g_{\Phi;\{w_1,w_2,w_3\}}(z) =$$
$$\frac{(w_1w_2\Phi + w_1w_2 - 1) \cdot z + (2w_1w_2w_3 - w_1 - w_3)\Phi + w_1w_2w_3}{w_2\Phi \cdot z + w_2w_3\Phi + w_2w_3 - 1}.$$

Further calculations suggest that the coefficients of $\Phi^1$ and $\Phi^0$ in these expressions are always a linear combinations of first-degree monomials $h$ in the $w_i$ such that the numerical coefficient of $h$ is $\pm 1$ times a Fibonacci number determined by the total degree of $h$; details are in the next section.

It is well known, of course, that the consecutive ratios $F_n/F_{n-1}$ of Fibonacci numbers converge to $\phi$. More generally, the limit of the ratios of consecutive elements of a linear recurrence $L$, when it exists, is called by Fiorenza and Vincenzi the *Kepler limit* of $L$. Certain roots of other polynomials than $x^2 - x - 1$ are also Kepler limits [5, 6], so we are led to consider the possibility that the $G_5$ phenomenon generalizes; our observations tend to confirm this guess. Section 2 describes what we found out about $G_5$, section 3 describes less detailed observations for $G_k, 5 \leq k \leq 33$, and the final section provides some detail about our numerical experiments; documentation in the form of *Mathematica* notebooks is here [4].

We state merely empirical claims in this article. When we say we have observed convergence of a sequence $s_n$ (say) to a limit $S$, we mean that our plots of 1000 values of $\log|S - s_n|$ are apparently linear, with negative slope. We relied on our eyesight in this matter: we did not fit our data to curves with a statistical package. Interested readers are invited to inspect the plots on our ResearchGate pages [4].

In the following section our observations were made on words in $W$ of length 20, and those in the last secton tested words of length 25. This means that we have in fact tested the claims on all shorter words as well.

In our tests, we identified functions in the $G_k$ with their matrix representations: A function

$$T_\lambda^{w_1} S T_\lambda^{w_2} S ... S T_\lambda^{w_n}$$

was identified with the corresponding matrix product.

More information about the Hecke groups is available, for example, in [3].

*Remark 1.*

The book [7] by Khovanskii apparently describes another method for approximating roots of polynomials using convergent sequences of ratios of elements of numerical sequences; but these sequences are not linear recurrences. (We have not seen [7], but Khovanskii's's method is described in [10], where the book is cited.)

## 2 The group $G_5$

We make the following definitions.

1. The Fibonacci numbers are defined with the convention that $F_0 = 0, F_1 = F_2 = 1$, *etc.* It will be convenient to write $F_{-1} = 1$ as well in contexts where (see below) $\overrightarrow{s} = \emptyset$.

2. $\chi$ is the following Dirichlet character:

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Alternatively, with $(a|b)$ representing the Kronecker symbol, $\chi(n) = (n|2)$ if $n \equiv 0, 1, 2, 3, 4$ or $6 \pmod 8$, and $\chi(n) = -(n|2)$ otherwise.

3. $W$ is the set of words $\overrightarrow{w}$ on $\mathbf{Z}$. The empty word $\overrightarrow{\emptyset}$ verifies $\overrightarrow{\emptyset} \in W$ and

$\overrightarrow{\emptyset} \subset \overrightarrow{w}$ for any $\overrightarrow{w} \in W$.

4. We write the cardinal number of a set $\sigma$ as $|\sigma|$. We apply the same notation to words $\overrightarrow{w} \in W$. We write $|\overrightarrow{\emptyset}| = 0$.

5. (a) For $\overrightarrow{w} \in W, \overrightarrow{w} \neq \overrightarrow{\emptyset}$, let $\overrightarrow{s} = \{w_{j_1}, w_{j_2}, ..., w_{j_m}\} \subset \overrightarrow{w} = \{w_1, ..., w_n\}$. If all of the $j_m \equiv m \pmod 2$,

then we write

$$\overrightarrow{s} \ll_1 \overrightarrow{w}.$$

We also write $\overrightarrow{\emptyset} \ll_1 \overrightarrow{w}$.

(b) If $\overrightarrow{s} \ll_1 \overrightarrow{w}$ and $|\overrightarrow{s}| > 1$, we write

$$\overrightarrow{s} \ll_2 \overrightarrow{w}.$$

(c) Let $\overrightarrow{s}, \overrightarrow{w}$ be as in definition 5a, except that all of the $j_m$ satisfy $j_m \equiv m-1 \pmod 2$. Then we write

$$\overrightarrow{s} \ll_3 \overrightarrow{w}.$$

We also write $\overrightarrow{\emptyset} \ll_3 \overrightarrow{w}$.

(d) If $\overrightarrow{s} \ll_3 \overrightarrow{w}$ and $|\overrightarrow{s}| > 1$, we write $\overrightarrow{s} \ll_4 \overrightarrow{w}$.

6. (a) For $\overrightarrow{w} \in W$, the formal product

$$m_{\overrightarrow{w}} = \prod_{w_i \in \overrightarrow{w}} w_i.$$

We also write

$$m_{\overrightarrow{\emptyset}} = 1.$$

(b) $M_{\overrightarrow{w}}$ is the set of all linear combinations with coefficients in the integers of monomials $m_{\overrightarrow{s}}$ such that $\overrightarrow{s} \subset \overrightarrow{w}$.

(c) $M$ is the union of the $M_{\overrightarrow{w}}$ as $\overrightarrow{w}$ ranges over $W$.

*Remark 2.*

In view of the identities $\Phi^2 = \Phi + 1$ for $\Phi = \phi$ or $\phi^*$, it is clear that

(i) For each $1 \leq j \leq 8$, there is a function $f_j : W \to M$ such that $f_j(\overrightarrow{w}) \in M_{\overrightarrow{w}}$ and, for all $g_{\Phi;\overrightarrow{w}} \in G(\Phi)$ and $\Im z > 0$,

$$g_{\Phi;\overrightarrow{w}}(z) = \frac{(f_1(\overrightarrow{w})\Phi + f_2(\overrightarrow{w}))z + f_3(\overrightarrow{w})\Phi + f_4(\overrightarrow{w})}{(f_5(\overrightarrow{w})\Phi + f_6(\overrightarrow{w}))z + f_7(\overrightarrow{w})\Phi + f_8(\overrightarrow{w})}.$$

Referring to the introduction, for example:

$$f_3(w_1, w_2, w_3) = 2w_1w_2w_3 - w_1 - w_3$$

and

$$f_6(w_1, w_2, w_3) = 0.$$

(ii) For each $1 \leq j \leq 8$, there is a function $\nu_j \colon W \times W \mapsto \mathbf{Z}$ determined by the condition

$$f_j(\overrightarrow{w}) = \sum_{\overrightarrow{s} \subset \overrightarrow{w}} \nu_j(\overrightarrow{s}, \overrightarrow{w}) m_{\overrightarrow{s}}.$$

The following observations describe the functions represented by words of length $\leq 20$ in $G_k, 5 \leq k \leq 50$.

*Observation 1.*

(i)

$$\nu_1(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|} & \text{if } \overrightarrow{s} \ll_1 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$\nu_2(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_1 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(iii)

$$\nu_3(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} -\chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|} & \text{if } \overrightarrow{s} \ll_1 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(iv)

$$\nu_4(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} -\chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_2 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(v)

$$\nu_5(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_3 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(vi)

$$\nu_6(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_4 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(vii)

$$\nu_7(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|} & \text{if } \overrightarrow{s} \ll_3 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(viii)

$$\nu_8(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_3 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

# 3 Higher-order Hecke groups

Definition: let $t(x)$ be a polynomial $\sum_{j=0}^{d} a_j x^j$ and $\gamma(t) := \gcd\{j \text{ s.t. } a_j \neq 0\}$. If $\gamma(t) = 1$, we say that $t$ is *stable* . Whether or not $t$ is stable, we associate to it the family of $d^{th}$-order linear recurrences $\Lambda_t$ with kernel $\{-a_{d-1}, -a_{d-2}, ..., -a_0\}$. Let $\lambda = 2\cos\frac{\pi}{k}$ with minimal polynomial $p_\lambda$. Under certain conditions [5, 6], a root $x = \kappa_\lambda$ of $p_\lambda(x)$ is the Kepler limit of one of the $L_\lambda \in \Lambda_{p_\lambda}$. The elements of $G(\lambda) = G_k$ have the form

$$g_{\lambda;\overrightarrow{w}}(z) = \frac{(\sum_{j=0}^{d-1} f_{\lambda,1,j}(\overrightarrow{w})\lambda^j) \cdot z + \sum_{j=0}^{d-1} f_{\lambda,2,j}(\overrightarrow{w})\lambda^j}{(\sum_{j=0}^{d-1} f_{\lambda,3,j}(\overrightarrow{w})\lambda^j) \cdot z + \sum_{j=0}^{d-1} f_{\lambda,4,j}(\overrightarrow{w})\lambda^j} \tag{1}$$

(Equation (1) is clear, as in the $G_5$ case, by substitution.)

For pragmatic reasons, we restricted our attention to $f = f_{\lambda,1,0}$ in the following observations.

*Observation 2.*

For $5 \leq k \leq 500, \gamma(p_\lambda) = 1$ if $k$ is odd, $\gamma(p_\lambda) = 2$ if $k$ is even.

*Observation 3.*

Let $5 \leq k \leq 33$.

(a) There is a function $\nu^{(k)} : W \times W \mapsto \mathbf{Z}$ such that

$$f(\overrightarrow{w}) = \sum_{\overrightarrow{s} \subset \overrightarrow{w}} \nu^{(k)}(\overrightarrow{s}, \overrightarrow{w}) m_{\overrightarrow{s}}. \tag{2}$$

and

$$\overrightarrow{s_1}, \overrightarrow{s_2} \subset \overrightarrow{w} \text{ and } |\overrightarrow{s_1}| = |\overrightarrow{s_2}| \Rightarrow |\nu^{(k)}(\overrightarrow{s_1}, \overrightarrow{w})| = |\nu^{(k)}(\overrightarrow{s_2}, \overrightarrow{w})| \tag{3}$$

for all $\overrightarrow{w} \in W$ s.t. $|\overrightarrow{w}| = 25$.

(b) If $k$ is odd, then for some particular $L_\lambda \in \Lambda_\lambda$ and all $\overrightarrow{s} \subset \overrightarrow{w}$ s.t. $|\overrightarrow{w}| = 25$:

(b1) $|\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})| \in L_\lambda$ and (b2) $\kappa_\lambda = \lambda$.

(In our experiments the sum on the r.h.s. of equation (2) typically contains over $6 \times 10^4$ terms, but twelve or fewer distinct values of $|\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})|$.)

(c) Suppose $6 \leq k \leq 32$ is even but $k \neq 14, 22$ or $26$. Then

(c1) clause (b1) still holds, but (b2) does not; in this situation, we found no $L_\lambda$ for which $\kappa_\lambda$ exists. (By design, our searches stopped with the first instance of $L_\lambda$ satisfying (a), so this is far from decisive.)

(c2) For $k = 8, 10, 16, 18$, and 32, the ratios of odd- and even-index members of the $L_\lambda$ we found form convergent sub-sequences with different limits.

(c3) For $k = 6, 12, 20, 24, 28$, and 30, the $L_\lambda$ terminate in a sequence in which alternate members are zero, so that the requisite ratios are alternately zero or undefined.

(d) Suppose $k = 12, 20, 24, 28$, or 30. After a substitution $y = x^2$, $p_\lambda(x)$ is transformed to a stable polynomial $q_{\lambda^2}(y)$ (say), and then $\lambda^2$ is the Kepler limit of a linear recurrence in $\Lambda_{q_{\lambda^2}}$.

(e) For $k = 14, 22$, or 26, we did not find $L_\lambda$ satisfying clause (a). Again (because we do not have a theoretical reason to rule out the existence of such linear recurrences) this is not decisive.

The conditions on polynomials $p$ under which a linear recurrence can be attached to $p$ with a Kepler limit that is killed by $p$ were established in [11] (cited in [5]).

A procedure (which can be invoked from computer algebra systems) for computing $p_\lambda$ for $\lambda = 2\cos\frac{\pi}{k}$ one at a time for individual $k$ has appeared in [2]; some information about the constant terms, in [1]; and, about the degree, in [8].

# References


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